

Quintic surface over p -adic local fields with infinite p -primary torsion in the Chow group of 0-cycles

Masanori Asakura

1 Introduction

In the paper [1] S. Saito and the author constructed a surface X over a p -adic local field such that the l -primary torsion part $\mathrm{CH}_0(X)[l^\infty]$ of the Chow group of 0-cycles is infinite for $l \neq p$, which gave a counter-example to a folklore conjecture. The purpose of this paper is to show that there is such an example even for $l = p$:

Theorem 1.0.1 (Theorem 4.1.2) *There is a quintic surface $X \subset \mathbb{P}_{\mathbb{Q}_p}^3$ over \mathbb{Q}_p such that the p -primary torsion part $\mathrm{CH}_0(X \times_{\mathbb{Q}_p} K)[p^\infty]$ is not finite for arbitrary finite extension K of \mathbb{Q}_p .*

Our proof is comparable with that of [1], however a new difficulty appears in case $l = p$. Let us recall the outline of the proof of [1] briefly. It follows from the universal coefficient theorem on Bloch's higher Chow group that we have the exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_l/\mathbb{Z}_l \xrightarrow{i} \mathrm{CH}^2(X, 1; \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \mathrm{CH}^2(X)[l^\infty] \longrightarrow 0 \quad (1.0.1)$$

for any l (possibly $l = p$). The proof of [1] breaks up into two steps. We first showed that if X is generic then $\mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ contains only decomposable elements supported on hyperplane section (cf. Lem. 3.2.2 below). Next we showed that the *boundary map*

$$\partial : \mathrm{CH}^2(X, 1; \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \mathrm{Pic}(Y) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is surjective (modulo finite groups) for X which has a good reduction Y . Thus if Y contains primitive divisors, then $\mathrm{CH}^2(X, 1; \mathbb{Q}_l/\mathbb{Z}_l)$ contains indecomposable elements and hence the map i cannot be surjective.

The technique used in the former step works also in case $l = p$. On the other hand, in the latter step, we used the result of Sato-Saito [6], in which they proved a weak Mordell-Weil type theorem for Chow group mod l different from p . Unfortunately its mod p counterpart has not been obtained. Thus we cannot use the same technique as in [1] to show the surjectivity of the boundary map in case $l = p$.

Actually we do not need the surjectivity of ∂ to prove Theorem 1.0.1. It is enough to show that the corank of the image of ∂ is greater than one. To do this, we construct an indecomposable element in $\mathrm{CH}^2(X, 1; \mathbb{Z}/p^n\mathbb{Z})$ (never coming from $\mathrm{CH}^2(X, 1) \otimes \mathbb{Z}/p^n\mathbb{Z}$!).

The strategy is as follows. We consider a quintic surface $X \subset \mathbb{P}_{\mathbb{Q}_p}^3$ which contains an irreducible quintic curve C with four nodes. Let $\tilde{C} \rightarrow C$ be the normalization and $\{P_i, Q_i\}$ ($1 \leq i \leq 4$) the inverse images of the four nodes on C . Since the curve \tilde{C} is genus 2, there is a rational function f_n on \tilde{C} such that $\text{div}(f_n) \equiv \sum_{i=1}^4 r_i(P_i - Q_i) \pmod{p^n}$ (this is a simple application of the theorem of Mattuck [5] which asserts that the Jacobian $J(\tilde{C})(\mathbb{Q}_p)$ is isomorphic to \mathbb{Z}_p^2 modulo finite groups). Thus the pair (C, f_n) determines an element in $\text{CH}^2(X, 1; \mathbb{Z}/p^n\mathbb{Z})$. We then prove that its boundary is nontrivial (hence indecomposable) under some assumptions.

This paper is organized as follows. In §3, we give an axiomatic approach to the construction of surface with infinite p -primary torsion in the Chow group of 0-cycles. In §4, we will construct such a quintic surface over \mathbb{Q}_p . A technical difficulty appears in the calculations of the boundary map. There we will use Igusa's j -invariants of hyperelliptic curves of genus two, which we list in Appendix for the convenience of the reader.

2 Preliminaries

For an abelian group M we denote by $M[n]$ (resp. M/n) the kernel (resp. cokernel) of the multiplication by n . We denote the p -primary torsion by $M[p^\infty] = \bigcup_{n \geq 1} M[p^n]$. For schemes X and T over a base scheme S , we write $X(T) = \text{Mor}_S(T, X)$ the set of S -morphisms, and say $x \in X(T)$ a T -valued point of X . If $T = \text{Spec } R$, then we also write $X(R) = X(\text{Spec } R)$ and say $x \in X(R)$ a R -rational point.

For a regular scheme X , we denote by $Z_i(X) = Z^{\dim X - i}(X)$ the free abelian group of irreducible subvarieties of Krull dimension i .

2.1 K -cohomology and Gersten complex

Let X be a smooth variety over a field F . Let us denote by X^i the set of irreducible subvarieties of X of codimension i . We write the function field of Z by η_Z .

Let \mathcal{K}_i be the sheaf associated to a presheaf $U \mapsto K_i(U)$ where $K_i(U)$ is Quillen's K -theory. The Zariski cohomology group $H^\bullet(X, \mathcal{K}_i)$ is called the K -cohomology. We only concern with $H^1(X, \mathcal{K}_2)$. It has an explicit description by using the Gersten complex

$$K_2^M(\eta_X) \xrightarrow{d_2} \bigoplus_{\text{codim } D=1} \eta_D^\times \xrightarrow{d_1} Z^2(X). \quad (2.1.1)$$

Recall the maps d_1 and d_2 . We denote by (f, D) the image of an element $f \in \eta_D^\times$ via the canonical inclusion $\eta_D^\times \rightarrow \bigoplus_{\text{codim } D=1} \eta_D^\times$. Then the map d_2 (called the *tame symbol*) is defined as follows

$$d_2\{f, g\} = \sum_{\text{codim } D=1} \left((-1)^{\text{ord}_D(f)\text{ord}_D(g)} \frac{f^{\text{ord}_D(g)}}{g^{\text{ord}_D(f)}}|_D, D \right).$$

The map d_1 is defined in the following way. Let $\tilde{D} \rightarrow D$ be the normalization. To $f \in \eta_D^\times$, we associate the Weil divisor $\text{div}_{\tilde{D}}(f)$ on \tilde{D} . Letting $j : \tilde{D} \rightarrow D \hookrightarrow X$ be the composition,

$d_1(f, D)$ is defined to be $j_*(\operatorname{div}_{\bar{D}}(f))$. It is simple to check $d_1 d_2 = 0$. Tensoring (2.1.1) with \mathbb{Z}/n , one has a complex

$$K_2^M(\eta_X)/n \xrightarrow{d_2 \otimes \mathbb{Z}/n} \bigoplus_{\operatorname{codim} D=1} \eta_D^\times/n \xrightarrow{d_1 \otimes \mathbb{Z}/n} Z^2(X)/n \quad (2.1.2)$$

Then the Gersten conjecture (Quillen's theorem) tells that (2.1.2) gives rise to a flasque resolution of the sheaf \mathcal{K}_2/n and hence one has the canonical isomorphism

$$H^1(X, \mathcal{K}_2/n) \cong \operatorname{Ker}(d_1 \otimes \mathbb{Z}/n) / \operatorname{Im}(d_2 \otimes \mathbb{Z}/n) \quad (2.1.3)$$

for each $n \geq 0$. Hereafter we always identify the K -cohomology $H^1(X, \mathcal{K}_2/n)$ with the group in the right hand side of (2.1.3).

2.2 $\operatorname{CH}^2(X, 1)$ and K -cohomology

We denote by $\operatorname{CH}^i(X, j; G)$ Bloch's higher Chow group with coefficients in an abelian group G . We simply write $\operatorname{CH}^i(X, j) = \operatorname{CH}^i(X, j; \mathbb{Z})$ and $\operatorname{CH}^i(X) = \operatorname{CH}^i(X, 0)$. By [3] 2.5, we have the canonical isomorphism

$$\operatorname{CH}^2(X, 1; \mathbb{Z}/n) \cong H^1(X, \mathcal{K}_2/n) \quad (2.2.1)$$

for each $n \geq 0$. We will also identify $\operatorname{CH}^2(X, 1; \mathbb{Z}/n)$ with $H^1(X, \mathcal{K}_2/n)$ by the above isomorphism.

By the universal coefficients theorem on higher Chow group there is the exact sequence

$$0 \longrightarrow \operatorname{CH}^2(X, 1)/n \longrightarrow \operatorname{CH}^2(X, 1; \mathbb{Z}/n) \longrightarrow \operatorname{CH}^2(X)[n] \longrightarrow 0 \quad (2.2.2)$$

for $n \neq 0$. Putting $n = l^k$ and taking the inductive limit on k , one obtains (1.0.1). Suppose that n is prime to the characteristic of F . Then there is the regulator map

$$\operatorname{reg}_X : \operatorname{CH}^2(X, 1; \mathbb{Z}/n) \longrightarrow H_{\text{ét}}^3(X, \mathbb{Z}/n(2))$$

to the étale cohomology group. Let

$$\begin{aligned} NH_{\text{ét}}^3(X, \mathbb{Z}/n(2)) &:= \operatorname{Ker}(H_{\text{ét}}^3(X, \mathbb{Z}/n(2)) \longrightarrow H_{\text{ét}}^3(\eta_X, \mathbb{Z}/n(2))) \\ &= \operatorname{Im}\left(\bigoplus_{\operatorname{codim} Z=1} H_Z^3(X, \mathbb{Z}/n(2)) \longrightarrow H_{\text{ét}}^3(X, \mathbb{Z}/n(2))\right) \end{aligned}$$

where the second equality follows from the localization exact sequence of étale cohomology. The main theorem of Bloch-Ogus theory tells that the image of the regulator map coincides with the above:

$$\operatorname{CH}^2(X, 1; \mathbb{Z}/n) \xrightarrow{\sim} NH_{\text{ét}}^3(X, \mathbb{Z}/n(2)). \quad (2.2.3)$$

It follows from (2.2.2) and (2.2.3) that one has Bloch's exact sequence

$$0 \longrightarrow \operatorname{CH}^2(X, 1)/n \longrightarrow NH_{\text{ét}}^3(X, \mathbb{Z}/n(2)) \longrightarrow \operatorname{CH}^2(X)[n] \longrightarrow 0. \quad (2.2.4)$$

If F is a p -adic local field, the cohomology group $H_{\text{ét}}^\bullet(X, \mathbb{Z}/n(j))$ (resp. $H_{\text{ét}}^\bullet(X, \mathbb{Q}_l/\mathbb{Z}_l(j))$) is known to be finite (resp. of cofinite type). Hence the n -torsion part $\operatorname{CH}^2(X)[n]$ is finite and $\operatorname{CH}^2(X)[l^\infty]$ is of cofinite type for any n and l :

$$\operatorname{CH}^2(X)[l^\infty] \cong (\mathbb{Q}_l/\mathbb{Z}_l)^{\oplus r} + (\text{finite group}).$$

2.3 Boundary map

Let R be a discrete valuation ring with a prime element π . Put $K := R[\pi^{-1}]$ and $\mathbb{F} := R/\pi R$. Let $X_R \rightarrow \operatorname{Spec} R$ be a projective smooth scheme over R . Put $X_K := X_R \times_R K$ and $X_{\mathbb{F}} := X_R \times_R \mathbb{F}$. There is the *boundary map*

$$\partial : \operatorname{CH}^2(X_K, 1; \mathbb{Z}/n\mathbb{Z}) \longrightarrow \operatorname{Pic}(X_{\mathbb{F}})/n, \quad n \geq 0. \quad (2.3.1)$$

Let us recall the definition. We freely use the identifications (2.1.3) and (2.2.1). Let $\sum(f, D) \in \operatorname{Ker}(d_1 \otimes \mathbb{Z}/n)$ where D is an irreducible divisor on X_K and f is a rational function on D . Let D_R be the Zariski closure of D in X_R . Let $\tilde{D}_R \rightarrow D_R$ be the normalization and $j_D : \tilde{D}_R \rightarrow D_R \hookrightarrow X_R$ the composition. The cycle

$$Z = \sum j_{D*} \operatorname{div}_{\tilde{D}_R}(f) \in Z^2(X_R)/n = Z_1(X_R)/n$$

is supported on $X_{\mathbb{F}}$ since $d_2(\sum(f, D)) = 0$ in $Z^2(X_K)/n$. Thus one can consider it to be a divisor on $X_{\mathbb{F}}$ and hence it determines an element $[Z]$ of $\operatorname{Pic}(X_{\mathbb{F}})$. We then define

$$\partial(\sum(f, D)) := [Z] \in \operatorname{Pic}(X_{\mathbb{F}})/n. \quad (2.3.2)$$

It is simple to show that (2.3.2) is well-defined, namely it annihilates the image of $d_2 \otimes \mathbb{Z}/n$.

3 Surface containing a curve with nodes

Let K be a finite extension of \mathbb{Q}_p , R the ring of integers and \mathbb{F} the residue field. For a scheme V_R over R , we write $V_K := V_R \times_R K$ and $V_{\mathbb{F}} := V_R \times_R \mathbb{F}$.

3.1 Conditions (A) and (B)

Let $X_R \subset \mathbb{P}_R^3$ be a hypersurface which is smooth over R and $C_R \subset X_R$ a hyperplane section which is flat over R . Let $\pi_0 : \tilde{C}_R \rightarrow C_R$ be the normalization and $i : C'_R \rightarrow \tilde{C}_R$ a desingularization, i.e. C'_R is a regular arithmetic surface which is proper flat over R and $C'_K \xrightarrow{\sim} \tilde{C}_K$. Put $\pi := \pi_0 i$.

$$\begin{array}{ccc} C'_R & \xrightarrow{i} & \tilde{C}_R \\ & \searrow \pi & \downarrow \pi_0 \\ & & C_R \xrightarrow{\subset} X_R. \end{array}$$

Let

$$C_{\mathbb{F}} = \sum_{j=1}^N D_j, \quad \tilde{C}_{\mathbb{F}} = \sum_{j=1}^N \tilde{D}_j, \quad C'_{\mathbb{F}} = \sum_{j=1}^N D'_j + \sum_l E_l$$

be the irreducible decompositions such that $\pi_0(\tilde{D}_j) = D_j$, $i(D'_j) = \tilde{D}_j$ and E_l are exceptional curves (see Remark 3.1.2). Note that $C_{\mathbb{F}}$ and $\tilde{C}_{\mathbb{F}}$ are reduced schemes (Remark 3.1.3).

We consider the following two conditions on (X_R, C_R) .

(A) X_K satisfies the following.

There is a nonsingular scheme S over \mathbb{Q} and a morphism $X_S \rightarrow S$ which has a Cartesian diagram

$$\begin{array}{ccc} X_K & \longrightarrow & X_S \\ \downarrow & \square & \downarrow \\ \text{Spec } K & \longrightarrow & S. \end{array}$$

induced from an embedding $\mathbb{Q}(S) \hookrightarrow K$ such that the complexes

$$0 \longrightarrow H^{i,2-i} \longrightarrow H^{i-1,3-i} \otimes \Omega_S^1 \quad (i = 1, 2) \quad (3.1.1)$$

$$H^{2,0} \longrightarrow H^{1,1} \otimes \Omega_S^1 \longrightarrow H^{0,2} \otimes \Omega_S^2 \quad (3.1.2)$$

$$0 \longrightarrow H^{2,0} \otimes \Omega_S^1 \longrightarrow H^{1,1} \otimes \Omega_S^2 \quad (3.1.3)$$

induced from the Gauss-Manin connection are exact at the middle terms. Here we put $H^{i,j} = (R^j f_* \Omega_{X_S/S}^i)_{\text{prim}}$ the Hodge (i, j) -component of the primitive cohomology $H_{\text{dR}}^2(X_S/S)_{\text{prim}} := H_{\text{dR}}^2(X_S/S)/[H]$ with H a hyperplane section.

(B) C_R satisfies the following conditions (B-1) and (B-2).

(B-1) (1) C_R is an irreducible (hence integral) scheme (cf. Remark 3.1.3).

(2) C_K has singular points A_1, \dots, A_m which are K -rational nodes.

(3) $\pi^{-1}(A_i)$ consists of two K -rational points P_i and Q_i .

(4) $C_{\mathbb{F}}$ is not irreducible (hence so is neither $\tilde{C}_{\mathbb{F}}$ nor $C'_{\mathbb{F}}$, cf. Remark 3.1.2).

(B-2) There are $r_i \in \mathbb{Z}_p$ ($1 \leq i \leq m$) which satisfy the following.

(1) Let $J = J(C'_K)$ be the Jacobian variety of C'_K and $\text{AJ} : \text{CH}_0(C'_K)_{\deg=0} \xrightarrow{\sim} J(K)$ the Abel-Jacobi map. Then

$$\sum_{i=1}^m r_i \text{AJ}(P_i - Q_i) = 0 \text{ in } \varprojlim_n J(K)/p^n. \quad (3.1.4)$$

(2) Let $P_{i,R}$ and $Q_{i,R}$ be the Zariski closure of P_i and Q_i in C'_R respectively. We denote by $(-\cdot-)_{C'_R}$ the intersection pairing on the arithmetic surface C'_R (cf. [4] Chapter III). Then there is no solution $x_l \in \mathbb{Q}_p$ which satisfy all of the following equalities

$$\begin{cases} \sum_l x_l (E_l \cdot D'_k)_{C'_R} + \sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot D'_k)_{C'_R} = 0 & \forall k, \\ \sum_l x_l (E_l \cdot E_s)_{C'_R} + \sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot E_s)_{C'_R} = 0 & \forall s. \end{cases} \quad (3.1.5)$$

In other words, there are $q_k, q'_s \in \mathbb{Z}_p$ such that

$$\sum_k q_k (E_l \cdot D'_k)_{C'_R} + \sum_s q'_s (E_l \cdot E_s)_{C'_R} = 0, \quad \forall l \quad (3.1.6)$$

$$\sum_k q_k \left(\sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot D'_k)_{C'_R} \right) + \sum_s q'_s \left(\sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot E_s)_{C'_R} \right) \neq 0. \quad (3.1.7)$$

In case $C'_R = \tilde{C}_R$ (i.e. $\{E_l\} = \emptyset$), this simply means

$$\sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot D'_k)_{C'_R} \neq 0, \quad \exists k.$$

Remark 3.1.1 Since $P_{i,R}$ and $Q_{i,R}$ are R -sections, they meet $C'_\mathbb{F}$ at nonsingular points transversally. Thus we have

$$(P_{i,R} \cdot D)_{C'_R} = \begin{cases} 1 & P_{i,R} \cap D \neq \emptyset \\ 0 & P_{i,R} \cap D = \emptyset \end{cases}, \quad (Q_{i,R} \cdot D)_{C'_R} = \begin{cases} 1 & Q_{i,R} \cap D \neq \emptyset \\ 0 & Q_{i,R} \cap D = \emptyset \end{cases}$$

for any component $D \subset C'_\mathbb{F}$.

Remark 3.1.2 There is 1-1 correspondence between irreducible components of $C_\mathbb{F}$ and those of $\tilde{C}_\mathbb{F}$ (hence $C_\mathbb{F} = \sum_j \pi_0(\tilde{D}_j)$ is the irreducible decomposition). In fact, π_0 is a finite morphism (EGA IV 7.8) which is an isomorphism over the regular locus C_R^{reg} . Therefore $\pi_0^{-1}(C_\mathbb{F}^{\text{reg}}) \rightarrow C_\mathbb{F}^{\text{reg}}$ is an isomorphism and $\tilde{C}_\mathbb{F} - \pi_0^{-1}(C_\mathbb{F}^{\text{reg}})$ is a finite set of closed points. This implies that there is a 1-1 correspondence between generic points of $C_\mathbb{F}$ and those of $\tilde{C}_\mathbb{F}$.

Remark 3.1.3 One can see that $C_\mathbb{F}$ (and hence $\tilde{C}_\mathbb{F}$) are automatically reduced. In fact let (x, y, z, w) be homogeneous coordinates of $\mathbb{P}_\mathbb{F}^3$ such that $C_\mathbb{F}$ is defined by $w = 0$. Write the defining equation of $X_\mathbb{F}$ by $F(x, y, z) + wG(x, y, z, w)$. If F has a decomposition $F = F_1^2 F_2$, then the zero locus $\{F_1 = G = w = 0\}$ turns out to be a singular locus of $X_\mathbb{F}$. This contradicts with the assumption that $X_\mathbb{F}$ is smooth.

Remark 3.1.4 The curves $\{D_j\}_j$ are linearly independent in $\text{Pic}(X_\mathbb{F}) \otimes \mathbb{Q}$. To see it it is enough to show that the matrix $M = ((D_j \cdot D_k)_{X_\mathbb{F}})_{1 \leq j, k \leq N}$ is nondegenerate where $(-\cdot-)_{X_\mathbb{F}}$ denotes the intersection pairing on $X_\mathbb{F}$. Let e_j be the degree of D_j . By definition,

$$e_j = (D_j \cdot C_\mathbb{F})_{X_\mathbb{F}} = (D_j \cdot \sum_{k=1}^N D_k)_{X_\mathbb{F}} = D_j^2 + \sum_{k \neq j} (D_j \cdot D_k)_{X_\mathbb{F}}.$$

We claim $(D_j \cdot D_k)_{X_\mathbb{F}} = e_j e_k$ for $j \neq k$. In fact let $F(x, y, z) + wG(x, y, z, w)$ be the defining equation of $X_\mathbb{F}$ as before. Let $F = \prod_{j=1}^N F_j$ be the irreducible decomposition. Let $P \in D_j \cap D_k$. Since $X_\mathbb{F}$ is smooth, $G(P) \neq 0$. One has

$$\mathcal{O}_{X_\mathbb{F}, P} / (F_j, F_k) \cong \mathcal{O}_{\mathbb{P}^3, P} / (F_j, F_k, F + wG) \cong \mathcal{O}_{\mathbb{P}^2, P} / (F_j, F_k).$$

Therefore $(D_j \cdot D_k)_{X_\mathbb{F}} = (D_j \cdot D_k)_{\mathbb{P}^2} = e_j e_k$ by the theorem of Bézout. Now we have

$$(D_j \cdot D_k)_{X_\mathbb{F}} = \begin{cases} e_j(1 - \sum_{l \neq j} e_l) & j = k \\ e_j e_k & j \neq k \end{cases}$$

and hence

$$\det M = e_1 \cdots e_N (1 - \sum_j e_j)^{N-1} \neq 0.$$

3.2 Infinite p -primary torsion in the Chow group of 0-cycles

Theorem 3.2.1 *Let (X_R, C_R) satisfies (A) and (B). Then the corank of $\mathrm{CH}_0(X_K)[p^\infty]$ is nonzero (hence it is infinite).*

This follows from Bloch's exact sequence (2.2.4) and the following Lemmas 3.2.2 and 3.2.3.

Lemma 3.2.2 *The image of the regulator map*

$$\mathrm{CH}^2(X_K, 1) \otimes \mathbb{Q}_p \longrightarrow H^1(G_K, H_{\mathrm{\acute{e}t}}^2(\overline{X}, \mathbb{Q}_p(2)))$$

coincides with the image of the map $H^1(G_K, \mathbb{Q}_p(1)) \rightarrow H^1(G_K, H_{\mathrm{\acute{e}t}}^2(\overline{X}, \mathbb{Q}_p(2)))$ induced from the hyperplane class $\mathbb{Q}_p(1) \hookrightarrow H_{\mathrm{\acute{e}t}}^2(\overline{X}, \mathbb{Q}_p(2))$. In other words the map

$$\mathrm{CH}^2(X_K, 1) \otimes \mathbb{Q}_p \longrightarrow H^1(G_K, H_{\mathrm{\acute{e}t}}^2(\overline{X}, \mathbb{Q}_p(2))_{\mathrm{prim}})$$

is zero.

Proof. The condition (A) implies the above assertion (the proof is the same as in [1]). \square

Lemma 3.2.3 *The corank of the image of the boundary map*

$$\partial : \mathrm{CH}^2(X_K, 1; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \mathrm{Pic}(X_{\mathbb{F}}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

is greater than 1. In other words the image of the map

$$\overline{\partial} : \mathrm{CH}^2(X_K, 1; \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow (\mathrm{Pic}(X_{\mathbb{F}})/[H]) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

has nonzero corank where H is a hyperplane section.

Proof. The proof makes use of the condition (B). We first construct an element $\xi_n \in \mathrm{CH}^2(X_K, 1; \mathbb{Z}/p^n\mathbb{Z})$. By (B-2) (3.1.4), there is a rational function f_n on C'_K such that

$$\mathrm{div}_{C'_K}(f_n) = \sum_{i=1}^m r_i (P_i - Q_i) \pmod{p^n Z_0(C'_K)}$$

where $Z_0(C'_K)$ denotes the free abelian group of closed points on C'_K . Since

$$\pi_* \mathrm{div}_{C'_K}(f_n) = \sum_{i=1}^m r_i (A_i - A_i) = 0 \pmod{p^n Z_0(X_K)}$$

the pair (f_n, C_K) determines an element $\xi_n \in \text{CH}^2(X_K, 1; \mathbb{Z}/p^n\mathbb{Z})$. By replacing f_n with cf_n for some constant $c \in K^*$, we may assume that the support of $\text{div}_{C'_R}(f_n)$ does not contain the component D'_1 so that we have

$$\text{div}_{C'_R}(f_n) \equiv \sum_{i=1}^m r_i(P_{i,R} - Q_{i,R}) + \sum_{j>1} n_j D'_j + \sum_l m_l E_l \pmod{p^n Z_1(C'_R)}.$$

By definition of the boundary map

$$\partial(\xi_n) = \sum_{j>1} n_j [D_j] \quad \text{in } \text{Pic}(X_{\mathbb{F}})/p^n$$

where $[D_j]$ denotes the cycle class of the divisor D_j in $\text{Pic}(X_{\mathbb{F}})$ (cf. §2.3). Note that $\{[D_j]\}_{j \geq 1}$ are linearly independent in $\text{Pic}(X_{\mathbb{F}}) \otimes \mathbb{Q}$ (cf. Remark 3.1.4). Therefore it is enough to show that $\min\{\text{ord}_p(n_j)\}_j$ is bounded as $n \rightarrow +\infty$ (since $\text{ord}_p(0) := +\infty$ by convention, it implies that some n_j is nonzero). Since

$$(\text{div}_{C'_R}(f_n) \cdot D)_{C'_R} = 0$$

for any components D of $C'_{\mathbb{F}}$ and the intersection numbers on C'_R are integers ([4] III §3), one has

$$\begin{cases} \sum_j n_j (D'_j \cdot D'_k)_{C'_R} + \sum_l m_l (E_l \cdot D'_k)_{C'_R} + \sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot D'_k)_{C'_R} \equiv 0 & \forall k, \\ \sum_j n_j (D'_j \cdot E_s)_{C'_R} + \sum_l m_l (E_l \cdot E_s)_{C'_R} + \sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot E_s)_{C'_R} \equiv 0 & \forall s \end{cases}$$

modulo p^n . It follows from **(B-2)** (3.1.6) and (3.1.7) that we have

$$\begin{aligned} - \sum_j n_j \left(\sum_k q_k (D'_j \cdot D'_k)_{C'_R} + \sum_s q'_s (D'_j \cdot E_s)_{C'_R} \right) = \\ \sum_k q_k \left(\sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot D'_k)_{C'_R} \right) + \sum_s q'_s \left(\sum_{i=1}^m r_i (P_{i,R} - Q_{i,R} \cdot E_s)_{C'_R} \right) \neq 0 \end{aligned}$$

Here n_j may depend on n . However so does none of r_i , q_k , q'_s or the above intersection numbers. Therefore at least one n_j is nonzero and its p -adic order is bounded by that of the right hand side. This completes the proof. \square

Corollary 3.2.4 *Let (X_R, C_R) satisfies **(A)** and **(B)**. Then the corank of $\text{CH}_0(X_K \times_K L)[p^\infty]$ is nonzero for arbitrary finite extension L/K .*

Proof. It is enough to show that $X_L := X_K \times_K L$ satisfies both of the assertions in Lemmas 3.2.2 and 3.2.3. Since **(A)** is clearly satisfied for X_L , Lemma 3.2.2 holds. The assertion in Lemma 3.2.3 remains true if we replace X_K with X_L because of the commutative diagram

$$\begin{array}{ccc} \text{CH}^2(X_K, 1; \mathbb{Q}_p/\mathbb{Z}_p) & \xrightarrow{\bar{\partial}} & (\text{Pic}(X_{\mathbb{F}})/[H]) \otimes \mathbb{Q}_p/\mathbb{Z}_p \\ \downarrow & & \downarrow \\ \text{CH}^2(X_L, 1; \mathbb{Q}_p/\mathbb{Z}_p) & \xrightarrow{\bar{\partial}} & (\text{Pic}(X_{\mathbb{F}_L})/[H]) \otimes \mathbb{Q}_p/\mathbb{Z}_p. \end{array}$$

Here the right vertical arrow is injective modulo finite group. This completes the proof. \square

4 Construction of Quintic surface

In this section we construct $(X_{\mathbb{Z}_p}, C_{\mathbb{Z}_p})$ which satisfies the conditions **(A)** and **(B)** ($N = 2$, $m = 4$) in case $X_{\mathbb{Z}_p}$ is a quintic surface.

4.1 Setting

Let

$$t = (a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, d_I)_{I=(i_0, i_1, i_2, i_3)}$$

be the homogeneous coordinates of $\mathbb{P}_{\mathbb{Z}}^{43}$ where I runs over the multi-indices such that $i_k \geq 0$ and $i_0 + i_1 + i_2 + i_3 = 4$. Put

$$\begin{aligned} H(x, y, z, w, t) &:= \sum_I d_I x^{i_0} y^{i_1} z^{i_2} w^{i_3}, \\ G(x, y, z, t) &:= x^2(x-z)^2 L_1 + y^2(y-z)^2 L_2 + xy(x-z)(y-z) L_3 \end{aligned}$$

where

$$L_1 = a_0 x + a_1 y + a_2 z, \quad L_2 = b_0 x + b_1 y + b_2 z, \quad L_3 = c_0 x + c_1 y + c_2 z.$$

We then consider a quintic homogeneous polynomial

$$F(x, y, z, w, t) := G(x, y, z, t) + wH(x, y, z, w, t) \quad (4.1.1)$$

parametrized by t . For an open set $S \subset \mathbb{P}_{\mathbb{Z}}^{43}$ we put

$$X_S := \{(x, y, z, w) \times t \in \mathbb{P}_{\mathbb{Z}}^3 \times S \mid F(x, y, z, w, t) = 0\}$$

$$C_S := \{(x, y, z) \times t \in \mathbb{P}_{\mathbb{Z}}^2 \times S \mid G(x, y, z, t) = 0\} = X_S \cap \{w = 0\}.$$

We thus have a family of quintic surface containing a quintic curve which has 4-nodes at $(x, y, z) = (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)$:

$$\begin{array}{ccc} C_S & \xrightarrow{\quad} & X_S \\ & \searrow & \swarrow \\ & S. & \end{array} \quad (4.1.2)$$

Hereafter we take S to be an affine open set of $\mathbb{P}_{\mathbb{Z}}^{43}$ (which is of finite type over \mathbb{Z}) such that $X_S \rightarrow S$ is smooth. Let

$$\begin{cases} d_2(w) = -(a_0 + a_2)w^2 - (c_0 + c_2)w - (b_0 + b_2) \\ d_1(w) = a_0 w^3 - (a_1 - c_0)w^2 + (b_0 - c_1)w - b_1 \\ d_0(w) = (a_1 + a_2)w^3 + (c_1 + c_2)w^2 + (b_1 + b_2)w \end{cases}$$

$$\begin{cases} e_2(u) = -(a_0 + a_2) - (c_0 + c_2)u - (b_0 + b_2)u^2 \\ e_1(u) = a_0 - (a_1 - c_0)u + (b_0 - c_1)u^2 - b_1u^3 \\ e_0(u) = (a_1 + a_2)u + (c_1 + c_2)u^2 + (b_1 + b_2)u^3 \end{cases}$$

and

$$\begin{aligned} U_1 &:= \{[s_0 : s_1] \times w \in \mathbb{P}^1 \times \mathbb{A}^1 \mid d_2(w)s_1^2 + d_1(w)s_1s_0 + d_0(w)s_0^2 = 0\}, \\ U_2 &:= \{[t_0 : t_1] \times u \in \mathbb{P}^1 \times \mathbb{A}^1 \mid e_2(u)t_0^2 + e_1(u)t_0t_1 + e_0(u)t_1^2 = 0\} \end{aligned}$$

where $\mathbb{P}^1 = \text{Proj } \mathcal{O}(S)[x_0, x_1]$ and $\mathbb{A}^1 = \text{Spec } \mathcal{O}(S)[z]$. We glue U_1 and U_2 by identification

$$[t_0 : t_1] \times u = [s_1 : ws_0] \times w^{-1}$$

and obtain a scheme \tilde{C}_S . Put $s := s_1/s_0$ and $t := t_1/t_0$. Hereafter we simply denote the coordinates $[s_0 : s_1] \times w$ and $[t_0 : t_1] \times u$ by (s, w) and (t, w) respectively. There is a finite morphism $\tilde{C}_S \rightarrow \mathbb{P}_S^1$ of degree 2 given by $(s, w) \mapsto w$. The generic fiber of $\tilde{C}_S \rightarrow S$ is a nonsingular hyperelliptic curve of genus 2. There is the normalization $\pi_S : \tilde{C}_S \rightarrow C_S$ given by

$$\begin{aligned} (s, w) &\longmapsto (x, y, z) = (s(w - s), w - s, w - s^2) \\ (t, u) &\longmapsto (x, y, z) = (t(1 - t), u(1 - t), u - t^2). \end{aligned}$$

Let $A_1 = (0, 0, 1)$, $A_2 = (0, 1, 1)$, $A_3 = (1, 0, 1)$, $A_4 = (1, 1, 1)$ be the 4-nodes of C_S . We put $\pi_S^{-1}(A_i) = \{P_i, Q_i\}$:

$$\begin{cases} P_1(s, w) = (\alpha_1, \alpha_1) \\ Q_1(s, w) = (\alpha_2, \alpha_2) \end{cases} \quad a_2\alpha_i^2 + c_2\alpha_i + b_2 = 0, \quad (4.1.3)$$

$$\begin{cases} P_2(s, w) = (0, \beta_1) \\ Q_2(s, w) = (0, \beta_2) \end{cases} \quad (a_1 + a_2)\beta_i^2 + (c_1 + c_2)\beta_i + b_1 + b_2 = 0, \quad (4.1.4)$$

$$\begin{cases} P_3(t, u) = (0, \gamma_1) \\ Q_3(t, u) = (0, \gamma_2) \end{cases} \quad (b_0 + b_2)\gamma_i^2 + (c_0 + c_2)\gamma_i + a_0 + a_2 = 0, \quad (4.1.5)$$

$$\begin{cases} P_4(s, w) = (1, \delta_1) \\ Q_4(s, w) = (1, \delta_2) \end{cases} \quad (a_0 + a_1 + a_2)\delta_i^2 + (c_0 + c_1 + c_2)\delta_i + b_0 + b_1 + b_2 = 0. \quad (4.1.6)$$

We fix a regular affine scheme T of finite type over \mathbb{Z} and a generically finite morphism $T \rightarrow S$ such that P_i and Q_i become T -valued points of \tilde{C}_S . In other words, the function field $\mathbb{Q}(T)$ contains $\mathbb{Q}(S)$ and all of $\alpha_i, \dots, \delta_i$ in (4.1.3), \dots , (4.1.6). Put $\tilde{C}_T := \tilde{C}_S \times_S T$:

$$\begin{array}{ccc} \tilde{C}_T & \longrightarrow & \tilde{C}_S \\ P_i, Q_i \updownarrow & \square & \downarrow \\ T & \longrightarrow & S. \end{array}$$

Theorem 4.1.1 *There exists an embedding $\sigma : \mathcal{O}(T) \hookrightarrow \mathbb{Z}_p$ such that the pair $(X_{\mathbb{Z}_p}, C_{\mathbb{Z}_p}) = (X_S \times_{\sigma} \mathbb{Z}_p, C_S \times_{\sigma} \mathbb{Z}_p)$ satisfies (A) and (B).*

The rest of this section is devoted to prove Theorem 4.1.1. Combining the above with Corollary 3.2.4, we have

Theorem 4.1.2 (Theorem 1.0.1) *There exists a quintic surface $X_{\mathbb{Q}_p} \subset \mathbb{P}_{\mathbb{Q}_p}^3$ over \mathbb{Q}_p such that $\mathrm{CH}_0(X_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} K)[p^\infty]$ is infinite for arbitrary finite extension K/\mathbb{Q}_p .*

4.2 Condition (A)

Proposition 4.2.1 *Let X_S/S be as above. Then by shrinking $S_{\mathbb{Q}} := S \times_{\mathbb{Z}} \mathbb{Q}$ to a small open set if necessary the sequences (3.1.1), (3.1.2) and (3.1.3) are exact. Thus for any embedding $\mathbb{Q}(S) \hookrightarrow K$, $X_K := X_S \times_S K$ satisfies the condition (A).*

Proof. Let

$$R_S = \mathcal{O}(S_{\mathbb{Q}})[x, y, z, w] / \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial w} \right)$$

be the Jacobian ring of the quintic polynomial F (4.1.1). It follows from the theory of Jacobian rings that one has

$$R_S^{11-5i} \cong H^{i,2-i} := F^i / F^{i+1} H_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \mathbb{Q}$$

for $0 \leq i \leq 2$. Moreover the tangent space of S is canonically isomorphic to the homogeneous part I^5 of degree 5 of ideal

$$I = \langle x^2(x-z)^2, y^2(y-z)^2, xy(x-z)(y-z), w \rangle$$

of R_S and the Gauss-Manin connection can be identified with the dual of the ring product $R_S^\bullet \otimes I^5 \rightarrow R_S^{\bullet+5}$. Thus to show the exactness of (3.1.1), (3.1.2) and (3.1.3) it is enough to show the following 4-sequences are exact:

$$R_S^{1+5i} \otimes I^5 \longrightarrow R_S^{6+5i} \longrightarrow 0 \quad (i = 0, 1) \quad (4.2.1)$$

$$R_S^1 \otimes \bigwedge^2 I^5 \longrightarrow R_S^6 \otimes I^5 \longrightarrow R_S^{11} \quad (4.2.2)$$

$$R_S^6 \otimes \bigwedge^2 I^5 \longrightarrow R_S^{11} \otimes I^5 \longrightarrow 0 \quad (4.2.3)$$

All of them can be checked by direct calculation (with the aid of computer). \square

4.3 Condition (B)

Let $\sigma : \mathcal{O}(T) \rightarrow \mathbb{Q}_p$ be a ring homomorphism. Set $X_{\mathbb{Z}_p}^\sigma := X_S \times_\sigma \mathbb{Z}_p$, $X_{\mathbb{Q}_p}^\sigma := X_S \times_\sigma \mathbb{Q}_p$, $X_{\mathbb{F}_p}^\sigma := X_S \times_\sigma \mathbb{F}_p$ and similarly for C_S and \tilde{C}_S . Put $P_{i,\mathbb{Z}_p}^\sigma := P_i \times_\sigma \mathbb{Z}_p$, $Q_{i,\mathbb{Z}_p}^\sigma := Q_i \times_\sigma \mathbb{Z}_p$ and $P_{i,\mathbb{Q}_p}^\sigma := P_i \times_\sigma \mathbb{Q}_p$, $Q_{i,\mathbb{Q}_p}^\sigma := Q_i \times_\sigma \mathbb{Q}_p$.

We consider the following conditions on σ . Write $a^\sigma = \sigma(a)$.

- (i) The homomorphism σ is injective. In other words, $(a_1/a_0)^\sigma, \dots, (d_I/a_0)^\sigma$ are algebraically independent over \mathbb{Q} .
- (ii) $X_{\mathbb{F}_p}^\sigma$ is smooth over \mathbb{F}_p (hence $X_{\mathbb{Z}_p}^\sigma$ is smooth over \mathbb{Z}_p).
- (iii) $\tilde{C}_{\mathbb{F}_p}^\sigma$ has two irreducible components D_1 and D_2 .
- (iv) $\tilde{C}_{\mathbb{Z}_p}^\sigma$ is an integral regular scheme (hence $\tilde{C}_{\mathbb{Q}_p}^\sigma$ is a nonsingular hyperelliptic curve of genus 2).
- (v) There is a subset $\{i_1, i_2, i_3\} \subset \{1, 2, 3, 4\}$ such that

$$\begin{cases} P_{i,\mathbb{Z}_p}^\sigma \cap D_1 \neq \emptyset \\ P_{i,\mathbb{Z}_p}^\sigma \cap D_2 = \emptyset \end{cases} \quad \text{and} \quad \begin{cases} Q_{i,\mathbb{Z}_p}^\sigma \cap D_1 = \emptyset \\ Q_{i,\mathbb{Z}_p}^\sigma \cap D_2 \neq \emptyset \end{cases} \quad (4.3.1)$$

or

$$\begin{cases} P_{i,\mathbb{Z}_p}^\sigma \cap D_1 = \emptyset \\ P_{i,\mathbb{Z}_p}^\sigma \cap D_2 \neq \emptyset \end{cases} \quad \text{and} \quad \begin{cases} Q_{i,\mathbb{Z}_p}^\sigma \cap D_1 \neq \emptyset \\ Q_{i,\mathbb{Z}_p}^\sigma \cap D_2 = \emptyset \end{cases} \quad (4.3.2)$$

for any $i \in \{i_1, i_2, i_3\}$.

- (vi) Let $J^\sigma = J(\tilde{C}_{\mathbb{Q}_p}^\sigma)$ be the Jacobian variety and $\text{AJ} : \text{CH}_0(\tilde{C}_{\mathbb{Q}_p}^\sigma)_{\deg=0} \xrightarrow{\sim} J^\sigma(\mathbb{Q}_p)$ the Abel-Jacobi map. Let $\{i_1, i_2, i_3\}$ be as in (vi). We may assume that (4.3.1) holds by exchanging $P_{i,\mathbb{Z}_p}^\sigma$ and $Q_{i,\mathbb{Z}_p}^\sigma$ if necessary. Then $\text{AJ}(P_{i_1,\mathbb{Q}_p}^\sigma - Q_{i_1,\mathbb{Q}_p}^\sigma) - \text{AJ}(P_{i_3,\mathbb{Q}_p}^\sigma - P_{i_3,\mathbb{Q}_p}^\sigma)$ and $\text{AJ}(P_{i_2,\mathbb{Q}_p}^\sigma - Q_{i_2,\mathbb{Q}_p}^\sigma) - \text{AJ}(P_{i_3,\mathbb{Q}_p}^\sigma - P_{i_3,\mathbb{Q}_p}^\sigma)$ are linearly independent over \mathbb{Q}_p in $J^\sigma(\mathbb{Q}_p) \otimes \mathbb{Q} \cong \mathbb{Q}_p^2$.

Proposition 4.3.1 *There exists σ which satisfies all of the conditions (i), \dots , (vi).*

We shall give a proof of Proposition 4.3.1 in §4.4.

Proposition 4.3.2 *Suppose that σ satisfies all of the conditions (i), \dots , (vi). Then $(X_{\mathbb{Z}_p}, C_{\mathbb{Z}_p}) = (X_{\mathbb{Z}_p}^\sigma, C_{\mathbb{Z}_p}^\sigma)$ satisfies the condition (B).*

Proof. (B-1) is straightforward. We see (B-2). Since J is a 2-dimensional abelian variety over \mathbb{Q}_p , one has

$$J^\sigma(\mathbb{Q}_p) \cong \mathbb{Z}_p^2 + (\text{finite group})$$

by the theorem of Mattuck [5] (see also [7] Part II Ch.V §7, Corollary 4). Therefore there is a nontrivial relation

$$\sum_{k=1}^3 r_k \text{AJ}(P_{i_k, \mathbb{Q}_p}^\sigma - Q_{i_k, \mathbb{Q}_p}^\sigma) = 0 \quad \text{in } \varprojlim_n J^\sigma(\mathbb{Q}_p)/p^n \quad (r_i \in \mathbb{Z}_p) \quad (4.3.3)$$

which gives the condition **(B-2)** (1). We show that r_1, r_2 and r_3 satisfy **(B-2)** (2). Since $\tilde{C}_{\mathbb{Z}_p}^\sigma$ is regular, what we want to show is that

$$\sum_{j=1}^3 r_i (P_{i_k, \mathbb{Z}_p}^\sigma - Q_{i_k, \mathbb{Z}_p}^\sigma \cdot D_l) \tilde{C}_{\mathbb{Z}_p}^\sigma = \pm \sum_{j=1}^3 r_i \quad (4.3.4)$$

is nonzero (see Remark 3.1.1 for the above equality). Suppose $r_1 + r_2 + r_3 = 0$. It follows from (4.3.3) that $\text{AJ}(P_{i_1, \mathbb{Q}_p}^\sigma - Q_{i_1, \mathbb{Q}_p}^\sigma) - \text{AJ}(P_{i_3, \mathbb{Q}_p}^\sigma - Q_{i_3, \mathbb{Q}_p}^\sigma)$ and $\text{AJ}(P_{i_2, \mathbb{Q}_p}^\sigma - Q_{i_2, \mathbb{Q}_p}^\sigma) - \text{AJ}(P_{i_3, \mathbb{Q}_p}^\sigma - Q_{i_3, \mathbb{Q}_p}^\sigma)$ are not linearly independent in $J^\sigma(\mathbb{Q}_p) \otimes \mathbb{Q} \cong \mathbb{Q}_p^2$. This contradicts with **(vi)**. \square

Theorem 4.1.1 follows from Propositions 4.2.1, 4.3.1 and 4.3.2.

4.4 Proof of Proposition 4.3.1

For a smooth scheme V over \mathbb{Q}_p , we denote by V^{an} a topological space $V(\mathbb{Q}_p)$ endowed with the p -adic manifold structure (cf. [7] Part II Chapter III). For a smooth scheme V over $k \subset \mathbb{Q}_p$, we simply write $V^{\text{an}} = (V \times_k \mathbb{Q}_p)^{\text{an}}$.

We put

$$\begin{aligned} U_2 &= \{\sigma : \mathcal{O}(T) \rightarrow \mathbb{Z}_p \mid \sigma \text{ satisfies (ii)}\} \subset T^{\text{an}} \\ &\vdots \\ U_5 &= \{\sigma : \mathcal{O}(T) \rightarrow \mathbb{Z}_p \mid \sigma \text{ satisfies (v)}\} \subset T^{\text{an}}, \end{aligned}$$

and

$$\begin{aligned} F_1 &= \{\sigma : \mathcal{O}(T) \rightarrow \mathbb{Q}_p \mid \sigma \text{ satisfies (i)}\} \subset T^{\text{an}} \\ U_6 &= \{\sigma : \mathcal{O}(T) \rightarrow \mathbb{Q}_p \mid \sigma \text{ satisfies (vi)}\} \subset T^{\text{an}}. \end{aligned}$$

Our goal is to show $F_1 \cap U_2 \cap \cdots \cap U_6 \neq \emptyset$.

Lemma 4.4.1 F_1 is a dense subset of T^{an} . Namely for any open ball \mathbb{B} in T^{an} , one has $\mathbb{B} \cap F_1 \neq \emptyset$.

Proof. Easy. \square

For ring homomorphisms $\sigma, \tau : \mathcal{O}(T) \rightarrow \mathbb{Z}_p$, we say $\sigma \equiv \tau \pmod{p^m}$ if $x^\sigma \equiv x^\tau \pmod{p^n}$ for all $x \in \mathcal{O}(T)$.

Lemma 4.4.2 (1) Suppose $\sigma \equiv \tau \pmod{p}$. Then σ satisfies (ii) (resp. (iii), (v)) if and only if so does τ .

(2) Suppose $\sigma \equiv \tau \pmod{p^2}$. Then σ satisfies (iv) if and only if so does τ .
Thus U_2, U_3, U_4 and U_5 are open sets of T^{an} .

Proof. (1) is clear. We see (2). It is easy to see that $\tilde{C}_{\mathbb{Z}_p}^\sigma$ is connected. It is enough to see whether $\tilde{C}_{\mathbb{Z}_p}^\sigma$ is regular or not around singular points of $\tilde{C}_{\mathbb{F}_p}^\sigma$. Let x be a singular point of $\tilde{C}_{\mathbb{F}_p}^\sigma$. Let f_i be the local equations of the irreducible component D_i of $\tilde{C}_{\mathbb{F}_p}^\sigma$ around x . Let F_i be a lifting of f_i to characteristic zero. Then one can write the local equation of $\tilde{C}_{\mathbb{Z}_p}^\sigma$ as $F_1 F_2 + pG$. $\tilde{C}_{\mathbb{Z}_p}^\sigma$ is regular around x if and only if $G(x) \not\equiv 0 \pmod{p}$. Thus the assertion follows. \square

It is not difficult to construct $\sigma : \mathcal{O}(S) \rightarrow \mathbb{Z}_p$ which satisfies (ii), \dots , (v). Hence

Lemma 4.4.3 $U_2 \cap U_3 \cap U_4 \cap U_5 \neq \emptyset$.

Lemma 4.4.4 U_6 is a dense open set of T^{an} .

To prove this, we prepare some notations. Let $T_{\mathbb{Q}} := T \times_{\mathbb{Z}} \mathbb{Q}$ and $\tilde{C}_{S_{\mathbb{Q}}} := \tilde{C}_S \times_{\mathbb{Z}} \mathbb{Q}$. Let \mathcal{M}_2 be the moduli scheme of curves of genus 2 over \mathbb{Q} , and $\mathcal{C} \rightarrow \mathcal{M}_2$ the universal curve. Then there is a dominant morphism $S_{\mathbb{Q}} \rightarrow \mathcal{M}_2$ such that $\tilde{C}_{S_{\mathbb{Q}}} \cong S_{\mathbb{Q}} \times_{\mathcal{M}_2} \mathcal{C}$. Note that $T_{\mathbb{Q}}$ is a 8-dimensional variety and \mathcal{M}_2 is a 3-dimensional variety. Put $\tilde{C}_{T_{\mathbb{Q}}} := \tilde{C}_S \times_S T_{\mathbb{Q}}$:

$$\begin{array}{ccccc} \tilde{C}_{T_{\mathbb{Q}}} & \longrightarrow & \tilde{C}_{S_{\mathbb{Q}}} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ T_{\mathbb{Q}} & \longrightarrow & S_{\mathbb{Q}} & \longrightarrow & \mathcal{M}_2. \end{array}$$

Let $\mathcal{J} \rightarrow \mathcal{M}_2$ be the Jacobian of \mathcal{C} . The divisor $P_i - Q_i$ induces the morphism

$$f_i : T_{\mathbb{Q}} \longrightarrow \mathcal{J}. \quad (4.4.1)$$

We first prove that U_6 is an open set. Recall the theorem of Mattuck. Let \mathcal{G} be the Lie algebra bundle of \mathcal{J} over \mathcal{M}_2 . We endow \mathcal{G} with the p -adic topology, and denote it by \mathcal{G}^{an} . Then there is a subbundle $\Lambda \subset \mathcal{G}^{\text{an}}$ whose fiber is isomorphic to \mathbb{Z}_p^2 and a subgroup bundle $G \subset \mathcal{J}^{\text{an}}$ of finite index such that

$$\begin{array}{ccc} \Lambda & \xrightarrow[\sim]{\text{exp}} & G \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{G}^{\text{an}} & & \mathcal{J}^{\text{an}} \end{array} \quad (4.4.2)$$

where “exp” is the exponential map ([7] Part II Ch.V §7). In particular there is a p -adically continuous homomorphism

$$\varepsilon : \mathcal{J}^{\text{an}} \longrightarrow \Lambda \quad (4.4.3)$$

whose kernel and cokernel are finite. Clearly it is an open map on p -adic manifolds. Put $g_1 := f_{i_1} - f_{i_3}$, $g_2 := f_{i_2} - f_{i_3}$ and $g := g_1 \times g_2 : T_{\mathbb{Q}} \rightarrow \mathcal{J} \times \mathcal{J}$. We denote by g_i^{an} etc. the associated p -adic analytic map:

$$g_i^{\text{an}} : T^{\text{an}} \longrightarrow \mathcal{J}^{\text{an}}, \quad g^{\text{an}} : T^{\text{an}} \longrightarrow \mathcal{J}^{\text{an}} \times \mathcal{J}^{\text{an}}. \quad (4.4.4)$$

Letting

$$V = \{(v_1, v_2) \in \Lambda \times \Lambda \mid v_1 \wedge v_2 \neq 0\} \quad (4.4.5)$$

be a dense open set, we have

$$U_6 = (g^{\text{an}})^{-1}(\varepsilon \times \varepsilon)^{-1}(V).$$

This shows that U_6 is an open set.

Next we show that U_6 is a dense subset. The map g gives rise to the map

$$(g_*)_x : \tan(T_{\mathbb{Q}})_x \longrightarrow \bigoplus_{i=1}^2 \tan(\mathcal{J})_{g_i(x)} \quad (4.4.6)$$

of the Zariski tangent space at a point $x \in T_{\mathbb{Q}}$.

Claim 4.4.5 *There is a closed subset $Z \subsetneq T_{\mathbb{Q}}$ such that (4.4.6) is surjective for all $x \notin Z$.*

Proof. Changing the variable s with Y by $Y = 2d_2(w)s + d_1(w)$ one has the Weierstrass form of the hyperelliptic curve $C_{T_{\mathbb{Q}}}$:

$$Y^2 = f(w) = d_1(w)^2 - 4d_0(w)d_2(w) = a_0^2 w^6 + \cdots + b_1^2.$$

We want to show that the pull-back

$$\bigoplus_{i=1}^2 g_i^* \Omega_{\mathcal{J}/\mathcal{M}_2}^1 \longrightarrow \Omega_{T_{\mathbb{Q}}/\mathcal{M}_2}^1, \quad (\omega_1, \omega_2) \longmapsto g_1^* \omega_1 + g_2^* \omega_2 \quad (4.4.7)$$

of Kähler differentials is injective at the generic point of $T_{\mathbb{Q}}$. Evaluating $a_0 = 0$, one has a closed subscheme $T_0 \hookrightarrow T_{\mathbb{Q}}$ and

$$C_{T_{\mathbb{Q}}} \times_{T_{\mathbb{Q}}} T_0 : Y^2 = f_0(w) = v_0 w^5 - v_1 w^4 + \cdots - v_5, \quad (4.4.8)$$

$$\begin{cases} v_0 = 4a_2(a_1 + a_2) \\ v_1 = -((a_1 + c_0)^2 + 4a_2(c_0 + c_1 + c_2) + 4c_2(a_1 + a_2)) \\ \vdots \end{cases}$$

Still $T_0 \rightarrow \mathcal{M}_2$ is dominant. We show that the composition map

$$\bigoplus_{i=1}^2 g_i^* \Omega_{\mathcal{J}/\mathcal{M}_2}^1|_{T_0} \longrightarrow \Omega_{T_{\mathbb{Q}}/\mathcal{M}_2}^1|_{T_0} \longrightarrow \Omega_{T_0/\mathcal{M}_2}^1 \quad (4.4.9)$$

is bijective at the generic point of T_0 , which implies the injectivity of (4.4.7) and hence the desired assertion. Note that $\Omega^1_{\mathcal{J}/\mathcal{M}_2}$ is a locally free sheaf of rank 2 generated by invariant 1-forms $\frac{dw}{Y}$ and $w\frac{dw}{Y}$. One has

$$\begin{aligned} f_1^* \frac{dw}{Y} &= \frac{d\alpha_1}{2\alpha_1 d_2(\alpha_1) + d_1(\alpha_1)} - \frac{d\alpha_2}{2\alpha_2 d_2(\alpha_2) + d_1(\alpha_2)} \\ f_1^* w \frac{dw}{Y} &= \frac{\alpha_1 d\alpha_1}{2\alpha_1 d_2(\alpha_1) + d_1(\alpha_1)} - \frac{\alpha_2 d\alpha_2}{2\alpha_2 d_2(\alpha_2) + d_1(\alpha_2)} \\ f_2^* \frac{dw}{Y} &= \frac{d\beta_1}{d_1(\beta_1)} - \frac{d\beta_2}{d_1(\beta_2)} \\ f_2^* w \frac{dw}{Y} &= \frac{\beta_1 d\beta_1}{d_1(\beta_1)} - \frac{\beta_2 d\beta_2}{d_1(\beta_2)} \\ f_3^* \frac{dw}{Y} &= \frac{\gamma_1 d\gamma_1}{e_1(\gamma_1)} - \frac{\gamma_2 d\gamma_2}{e_1(\gamma_2)} \\ f_3^* w \frac{dw}{Y} &= \frac{d\gamma_1}{e_1(\gamma_1)} - \frac{d\gamma_2}{e_1(\gamma_2)} \\ f_4^* \frac{dw}{Y} &= \frac{d\delta_1}{2d_2(\delta_1) + d_1(\delta_1)} - \frac{d\delta_2}{2d_2(\delta_2) + d_1(\delta_2)} \\ f_4^* w \frac{dw}{Y} &= \frac{\delta_1 d\delta_1}{2d_2(\delta_1) + d_1(\delta_1)} - \frac{\delta_2 d\delta_2}{2d_2(\delta_2) + d_1(\delta_2)}. \end{aligned}$$

We want to show that

$$f_{i_1}^* \frac{dw}{Y} - f_{i_3}^* \frac{dw}{Y}, \quad f_{i_1}^* w \frac{dw}{Y} - f_{i_3}^* w \frac{dw}{Y}, \quad f_{i_2}^* \frac{dw}{Y} - f_{i_3}^* \frac{dw}{Y}, \quad f_{i_2}^* w \frac{dw}{Y} - f_{i_3}^* w \frac{dw}{Y} \quad (4.4.10)$$

generate $\Omega^1_{T_0/\mathcal{M}_2}$ as \mathcal{O}_{T_0} -module at the generic point.

The affine coordinate ring of \mathcal{M}_2 is described by Igusa's j -invariants J_2, J_4, J_6, J_8 and J_{10} ([2]). In particular, $\Omega^1_{\mathcal{M}_2/\mathbb{Q}}$ is generated by

$$d(J_4/J_2^2), \quad d(J_6/J_2^3), \quad d(J_{10}/J_2^5) \quad (4.4.11)$$

generically. Therefore it is enough to show that (4.4.10) and (4.4.11) generate $\Omega^1_{T_0/\mathbb{Q}}$ as \mathcal{O}_{T_0} -module at the generic point. We know the explicit forms of J_2, \dots, J_{10} (see §5 Appendix). Therefore one can check it by direct calculations (the details are left to the reader since they are long and tedious). \square

We prove that U_6 is dense in T^{an} . Let $\sigma \in T^{\text{an}}$ be an arbitrary point. For any open ball \mathbb{B} about σ , we want to show $\mathbb{B} \cap U_6 \neq \emptyset$. There is a point $\sigma_0 \in \mathbb{B} - Z(\mathbb{Q}_p)$. So it is enough to show $\mathbb{B}_0 \cap U_6 \neq \emptyset$ for any open ball \mathbb{B}_0 about σ_0 . It follows from Claim 4.4.5 that the map g^{an} (4.4.4) is an open map on (sufficiently small) \mathbb{B}_0 . In particular $g^{\text{an}}(\mathbb{B}_0)$ is an open subset of $\mathcal{J}^{\text{an}} \times \mathcal{J}^{\text{an}}$ and hence $(\varepsilon \times \varepsilon)g^{\text{an}}(\mathbb{B}_0)$ is also open in $\Lambda \times \Lambda$. Since V (4.4.5) is a dense open set, we have $V \cap (\varepsilon \times \varepsilon)g^{\text{an}}(\mathbb{B}_0) \neq \emptyset$. This means $\mathbb{B}_0 \cap U_6 = \mathbb{B}_0 \cap (g^{\text{an}})^{-1}(\varepsilon \times \varepsilon)^{-1}(V) \neq \emptyset$, which is the desired assertion. This completes the proof of Lemma 4.4.3.

We finish the proof of Proposition 4.3.1. It follows from Lemmas 4.4.3 and 4.4.4 that $U_2 \cap \cdots \cap U_6$ is a nonempty open set. By Lemma 4.4.1 one has $F_1 \cap U_2 \cap \cdots \cap U_6 \neq \emptyset$. This is the desired assertion.

5 Appendix: Igusa's j -invariants

In [2], Igusa gave the arithmetic invariants of hyperelliptic curves of genus 2. For a hyperelliptic curve which has an affine equation

$$y^2 = f(x) = v_0x^5 - v_1x^4 + v_2x^3 - v_3x^2 + v_4x - v_5$$

they are given as follows.

$$\begin{aligned} J_2 &= 5v_0v_4 - 2v_1v_3 + 4^{-1}3v_2^2 \\ J_4 &= -8^{-1}[25v_0^2v_3v_5 - 15v_0^2v_4^2 - 15v_0v_1v_2v_5 + 7v_0v_1v_3v_4 + 2^{-1}v_0v_2^2v_4 \\ &\quad - v_0v_2v_3^2 + 4v_1^3v_5 - v_1^2v_2v_4 - v_1^2v_3^2 + v_1v_2^2v_3 - 3 \cdot 2^{-4}v_4^4] \\ J_6 &= -16^{-1}[2^{-1}5^3v_0^3v_2v_5^2 - 25v_0^3v_3v_4v_5 + 5v_0^3v_4^3 - 25v_0^2v_1^2v_5^2 \\ &\quad - 10v_0^2v_1v_2v_4v_5 + 10v_0^2v_1v_3^2v_5 - v_0^2v_1v_3v_4^2 - 4^{-1}5v_0^2v_2^2v_3v_5 \\ &\quad - 4^{-1}11v_0^2v_2^2v_4^2 + 2^{-1}7v_0^2v_2v_3^2v_4 - v_0^2v_3^4 + 6v_0v_1^3v_4v_5 \\ &\quad - 3v_0v_1^2v_2v_3v_5 + 2^{-1}7v_0v_1^2v_2v_4^2 - 2v_0v_1^2v_3^2v_4 + 3 \cdot 4^{-1}v_0v_1v_2^3v_5 \\ &\quad - 4^{-1}7v_0v_1v_2^2v_3v_4 + v_0v_1v_2v_3^3 + 7 \cdot 16^{-1}v_0v_4^4v_4 - 4^{-1}v_0v_2^3v_3^2 - v_1^4v_4^2 \\ &\quad + v_1^3v_2v_3v_4 - 4^{-1}v_1^2v_2^3v_4 - 4^{-1}v_1^2v_2^2v_3^2 + 8^{-1}v_1v_2^4v_3 - 2^{-6}v_2^6] \\ J_8 &= 4^{-1}[J_2J_6 - J_4^2] \\ J_{10} &= v_0D \end{aligned}$$

where

$$D = \begin{vmatrix} v_0 & -v_1 & v_2 & -v_3 & v_4 & -v_5 & 0 & 0 & 0 \\ 0 & v_0 & -v_1 & v_2 & -v_3 & v_4 & -v_5 & 0 & 0 \\ 0 & 0 & v_0 & -v_1 & v_2 & -v_3 & v_4 & -v_5 & 0 \\ 0 & 0 & 0 & v_0 & -v_1 & v_2 & -v_3 & v_4 & -v_5 \\ 5v_0 & -4v_1 & 3v_2 & -2v_3 & v_4 & 0 & 0 & 0 & 0 \\ 0 & 5v_0 & -4v_1 & 3v_2 & -2v_3 & v_4 & 0 & 0 & 0 \\ 0 & 0 & 5v_0 & -4v_1 & 3v_2 & -2v_3 & v_4 & 0 & 0 \\ 0 & 0 & 0 & 5v_0 & -4v_1 & 3v_2 & -2v_3 & v_4 & 0 \\ 0 & 0 & 0 & 0 & 5v_0 & -4v_1 & 3v_2 & -2v_3 & v_4 \end{vmatrix}$$

is Sylvester's resultant.

Putting the degree of J_{2i} to be $2i$, the affine coordinate ring of \mathcal{M}_2 is given by the homogeneous part of degree 0 in the graded ring $\mathbb{Q}[J_2, \dots, J_{10}, J_{10}^{-1}]$ ([2] Theorem 2).

References

- [1] Asakura, M., Saito, S.: Surfaces over a p -adic field with infinite torsion in the Chow group of 0-cycles. *Algebra Number Theory* **1**, 163–181 (2007)

- [2] Igusa, J.: Arithmetic variety of moduli for genus two. *Ann. of Math.* (2) 72 1960 612–649.
- [3] S. Landsburg: *Relative Chow groups*, *Illinois J. of Math.* **35** (1991), 618–641.
- [4] Lang, S.: *Introduction to Arakelov theory*, Springer, 1988.
- [5] Mattuck, A.: Abelian varieties over p -adic ground fields. *Ann. of Math.* (2) 62, (1955). 92–119.
- [6] Saito, S., Sato, K.: Finiteness theorem on zero-cycles over p -adic fields. to appear in *Ann. of Math.*,
- [7] Serre, J-P.: *Lie algebras and Lie groups*. 1964 lectures given at Harvard University. Second edition. *Lecture Notes in Mathematics*, 1500. Springer.

Department of Mathematics, Hokkaido University, Sapporo 060-0810, JAPAN

E-mail : **asakura@math.sci.hokudai.ac.jp**